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Subalgebras, Intervals, and Central Elements of Generalized Effect Algebras

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The relation between generalized effect algebras and D-algebras and their subalgebras are discussed. For generalized effect algebras the notion of central elements is introduced and some of their properties are shown.

INTRODUCTION

In the axiomatic approach to quantum mechanics the event structure of a physical system is identified with a quantum logic as an orthomodular lattice or poset (Varadarajan, 1968; Kalmbach, 1983; Beran, 1984; Pták and Pulmannová 1991). In recent years, new algebraic structures weakening the axiomatic system of orthomodular lattices (or posets) have been introduced for investigations in the foundations of quantum mechanics. Some of them are equivalent in some sense, or they are in some close connection. The analysis of all of them is not the subject of this note. We will discuss only relationships between D-posets, D-algebras, and effect algebras and their generalized versions.

Kôpka (1992) introduced a new algebraic structure of fuzzy sets, a D-poset of fuzzy sets. A difference of comparable fuzzy sets is a primary operation in this structure. Later, Kôpka and Chovanec (1994), by transferring the properties of a difference operation of a D-poset of fuzzy sets to an arbitrary partially ordered set, obtained a new algebraic structure, a D-poset that generalizes orthoalgebras and MV algebras.

Events of quantum logics do not describe "unsharp measurements," since unsharp measurements do not have a "yes–no" character. To include

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such events another algebraic structure was introduced by Foulis and Bennett (1994), called an effect algebra. Unsharp orthoalgebras have been defined by Giuntini and Greuling (1989).

Unbounded versions of effect algebras and D-posets (more precisely, not necessary bounded posets) have also been considered (Foulis and Bennett 1994; Kôpka and Chovanec, 1994; Kalmbach and Riečanová, 1994; Hédlíková and Pulmannová, 1996).

1. EFFECT ALGEBRAS, *D*-POSETS, AND *D*-ALGEBRAS

In the remainder, for a partial operation $\oplus$ (or $\ominus$) on a set X and for a, b, $c \in X$ if we write $a \oplus b = c$ ($c \ominus b = a$) we mean both that $a \oplus b$ ($c \ominus b$) is defined and $a \oplus b = c$ ($c \ominus b = a$).

Definition 1.1 (Kôpka and Chovanec, 1994). Let $(P, \leq)$ be a poset with the least element 0 and the greatest element 1. Let $\ominus$ be a partial binary operation on P such that $b \ominus a$ is defined iff $a \leq b$. Then $(P; \leq, \ominus, 0, 1)$ is called a *difference poset* (D-poset) if the following conditions are satisfied:

(Di) For any $a \in P$, $a \ominus 0 = a$.
(Dii) If $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

Effect algebras (introduced by Foulis and Bennett, 1994) are important for modeling unsharp measurements in Hilbert space: The set of all effects is the set of all self-adjoint operators T on a Hilbert space H with $0 \leq T \leq 1$. In a general algebraic form an effect algebra is defined as follows:

Definition 1.2. A structure $(E; \oplus, 0, 1)$ is called an *effect algebra* if 0, 1 are two distinguished elements and $\oplus$ is a partially defined binary operation on P which satisfies the following conditions for any $a, b, c \in E$:

(Ei) $b \oplus a = a \oplus b$ if $a \oplus b$ is defined.
(Eii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if one side is defined.
(Eiii) For every $a \in P$ there exists a unique $b \in P$ such that $a \oplus b = 1$.
(Eiv) If $1 \oplus a$ is defined, then $a = 0$.

We can easily show the following statement.

Proposition 1.3 (Cancellation properties)
(a) In a D-poset $(P; \leq, \ominus, 0, 1)$ if $a \leq b$ and $a \leq c$ for $a, b, c \in P$, then

$$b \ominus a = c \ominus a \quad \text{implies} \quad b = c$$

(b) In an effect algebra $(E; \oplus, 0, 1)$ if $a \oplus b$ and $a \oplus c$ are defined for $a, b, c, \in E$, then

$$a \oplus b = a \oplus c \quad \text{implies} \quad b = c$$

Corollary 1.4. (1) In every D-poset $(P; \leq, \ominus, 0, 1)$ the partial binary operation $\oplus$ can be defined by

(EP) $a \oplus b$ is defined and $a \oplus b =$ c iff $a \leq c$ and $c \ominus a = b$.

(2) In every effect algebra E the partial binary operation $\ominus$ and the relation $\leq$ can be defined by

(PE) $a \leq c$ and $c \ominus a = b$ iff $a \oplus b$ is defined and $a \oplus b = c$.

Cancellation properties guarantee that $\oplus$, $\ominus$, and $\leq$ are well defined. Moreover, it is easy to show the following statements:

Proposition 1.5. (1) In every D-poset $(P: \leq, \ominus, 0, 1)$ the partial binary operation $\oplus$ derived by (EP) fulfills axioms (Ei)–(Eiv) of an effect algebra.

(2) In every effect algebra $(E; \oplus, 0, 1)$ the partial binary operation $\ominus$ and the partial order $\leq$ defined by (PE) fulfills axioms (Di)–(Dii) of a D-poset.

A D-algebra is a generalization of a D-poset in which a partial order is not assumed. However, if a D-algebra is equipped with a natural partial order (derived from the partial operation $\ominus$), then it becomes a D-poset. We present here the definition of D-algebra by Gudder (1994).

Definition 1.6. A partial algebra $(P; \ominus, 0, 1)$ is called a *D-algebra* if 0, 1 are two distinguished elements of P and $\ominus$ is a partially defined binary operation on P which satisfies the following conditions for any $a, b, c \in P$:

(Ai) $a \ominus 0$ is defined and $a \ominus 0 = a$ for all $a \in P$.
(Aii) $1 \ominus a$ is defined for all $a \in P$.
(Aiii) If $0 \ominus a$ is defined, then $a = 0$.
(Aiv) If $b \ominus a$ and $c \ominus b$ is defined, then $c \ominus a$ and $(c \ominus a) \ominus (c \ominus b)$ are defined and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

We can easily show the following statements:

Proposition 1.7. (i) If $(P; \leq, \ominus, 0, 1)$ is a D-poset, then $(P; \ominus, 0, 1)$ is a D-algebra.

(ii) If $(P; \ominus, 0, 1)$ is a D-algebra and we define $a \leq b$ iff $b \ominus a$ is defined, then the relation $\leq$ is a partial order on P and $(P; \leq, \ominus\ 0, 1)$ is a D-poset.

See Gudder (1994) for the proof.

Corollary 1.8. In every. D-algebra $(P; \oplus, 0, 1)$ for $a, b, c \in P$ the cancellation property $b \ominus a = c \ominus a$ implies $b = c$ is satisfied and the partial binary operation $\oplus$ can be defined by

(EA) $a \oplus b$ is defined and $a \oplus b = c$ iff $c \ominus a$ is defined and $c \ominus a = b$.

Then $(P; \oplus, 0, 1)$ is an effect algebra.

2. GENERALIZED EFFECT ALGEBRAS, *D*-POSETS, AND *D*-ALGEBRAS

Generalizations of effect algebras, *D*-posets, and *D*-algebras have been studied by Kôpka and Chovanec (1994) (difference posets), Foulis and Bennett, 1994 (cones), Kalmbach and Riečanová (1994) (abelian *RI*-posets and abelian *RI* semigroups), and Hedlíková and Pulmannová (1996) (generalized *D*-posets and cancelative positive partial abelian semigroups). Hedlíková and Pulmannová (1996) proved that every generalized *D*-poset is an order ideal of a suitable *D*-poset [thus extending previous similar results for generalized Boolean algebras, results of Janowitz for generalized orthomodular lattices, and of Mayet-Ippolito for (weak) generalized orthomodular posets]. It can be shown that all of the above-mentioned generalizations of effect algebras (cones, abelian *RI*-semigroups, cancelative positive PAS) are mutually equivalent algebraic structures and can be derived from generalized *D*-posets (deriving $\oplus$ from $\ominus$ similarly as for deriving effect algebra from *D*-poset) and we will call them all generalized effect algebras. Thus their common definition is the following:

Definition 2.1. A partial algebra $(E; \oplus, 0)$ is called a *generalized effect algebra* if $0 \in E$ is a distinguished element and $\oplus$ is a partially defined binary operation E which satisfies the following conditions for any $a, b, c \in E$:

(GEi) $a \oplus b = b \oplus a$, if one side is defined.
(GEii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$, if one side is defined.
(GEiii) $a \oplus 0 = a$ for all $a \in E$.
(GEiv) $a \oplus b = a \oplus c$ implies $b = c$ (cancellation law).
(GEv) $a \oplus b = 0$ implies $a = b = 0$.

Definition 2.2. A partial algebra $(P; \ominus, 0)$ is called a *generalized D-algebra* (*D-poset*) if $0 \in P$ is a distinguished element and $\ominus$ is a partially defined binary operation on P which satisfies the following conditions for any $a, b, c \in P$:

(GDi) $a \ominus 0 = a$ for all $a \in P$.
(GDii) $a \ominus a = 0$ for all $a \in P$.
(GDiii) If $b \ominus a$ is defined, then $b \ominus (b \ominus a)$ is defined.
(GDiv) $(a \ominus b) \ominus c = (a \ominus c) \ominus b$, if one side is defined.

(GDv) If $b \ominus a$ and $c \ominus b$ are defined, then $c \ominus a$ is defined.
(GDvi) If $a \ominus b$ and $b \ominus a$ are defined, then $a = b$.
(GDvii) $c \ominus a = b \ominus a$ implies $c = b$ (cancellation law).

Proposition 2.3. (1) In every generalized D-algebra $(P; \ominus, 0)$ the partial binary operation $\oplus$ can be defined by

(GED) $a \oplus b$ is defined and $a \oplus b = c$ iff $c \ominus a$ is defined and $c \ominus a = b$ and the partial order in P can be defined by $a \leq b$ iff $b \ominus a$ is defined.

(2) In every generalized effect algebra $(E; \oplus, 0)$ the partial binary operation $\ominus$ can be defined by

(GDE) $a \ominus b$ is defined and $a \ominus b = c$ iff $b \oplus c$ is defined and $b \oplus c = a$ and the partial order in P can be defined by $a \leq b$ iff there exists $c \in P$ with $a \oplus c = b$.

Cancellation laws (GEiv) and (GDvii) guarantee that $\oplus$, $\ominus$, and $\leq$ are well defined. Moreover, we can show the following statements:

Proposition 2.4. (1) In every generalized effect algebra $(E; \oplus, 0)$ the partial binary operation $\ominus$ derived by (GDE) fulfills axioms (GDi)–(GDvii) of a generalized D-algebra.

(2) In every generalized D-algebra $(P; \ominus, 0)$ the partial binary operation $\oplus$ derived by (GED) fulfills axioms of a generalized effect algebra.

Moreover, the partial orders derived from the corresponding operations $\oplus$ and $\ominus$, i.e., derived one from the other by (GED), respectively (GDE), coincide.

Proposition 2.5. (1) If $(P; \ominus, 0)$ is a generalized D-algebra and there is an element $1 \in P$ such that $1 \ominus a$ is defined for all $a \in P$, then $(P; \ominus, 0, 1)$ is a D-algebra (and $(P; \leq, \ominus, 1)$ with $\leq$ derived by $\ominus$ is a D-poset).

(2) If $(P; \oplus, 0)$ is a generalized effect algebra and there is $1 \in P$ such that for all $a \in P$ there is $b \in P$ with $a \oplus b = 1$, then $(P; \oplus, 0, 1)$ is an effect algebra.

Proofs of Propositions 2.3–2.5 can be found in Hedelíková and Pulmanová (1996) and Kalmbach and Riečanová (1994). Let us mention that a *generalized D-poset* is a generalized D-algebra with partial order derived from the operation $\ominus$. By the next proposition every generalized D-algebra (every generalized effect algebra) can be embedded into a D-algebra (effect algebra).

In the following proposition let us denote by $(P^*: \leq_{P^*})$ the dual poset to a poset $(P; \leq_P)$ and by a^* the element of P^* corresponding to $a \in P$. Hence, for $a^*, b^* \in P^*$ we have $a^* \leq_{P^*} b^*$ iff $b \leq_P a$.

Proposition 2.6 (Hedlíková and Pulmannová, 1996). Let $(P; \ominus_P, 0_P)$ be a generalized D-algebra and let $\oplus$ be derived from $\ominus_P$ by (GED). Let $a \leq_P b$ iff $b \ominus_P a$ is defined and let $(P^*; \leq_{P^*})$ be the dual poset to $(P; \leq_P)$. Suppose that $\mathbb{P} = P \cup P^*$ is disjoint union and define the partial order $\leq$ and the partial binary operation $\ominus$ on $\mathbb{P}$ by the following conditions:

(1) For $a, b \in P$, $a \leq b$ iff $a \leq_P b$ and then we put $b \ominus a = b \ominus_P a$.

(2) For $a^*, b^* \in P^*$, $a^* \leq b^*$ iff $b \leq_P a$ and then we put $b^* \ominus a^* = a \ominus_P b$.

(3) For $a \in P$, $b^* \in P^*$, $a \leq b^*$ iff $b \leq a^*$ iff $a \oplus_P b$ is defined and then we put $b^* \ominus a = a^* \ominus b = (a \oplus_P b)^*$.

(4) For all $a \in P$ and $b^* \in P^*$, $a \ominus b^*$ is not defined (hence, $b^* \leq a$ does not hold).

Then $(\mathbb{P}; \ominus, 0, 1)$, where $0 = 0_P$, $1 = 0_P^*$, is a D-algebra. Moreover, $(P, \leq_P)$ is an order ideal in $(\mathbb{P}; \leq)$.

Suppose now that $(P; \oplus_P, 0_P)$ is a generalized effect algebra. Let $(P; \ominus_P, 0_P)$ be a generalized D-algebra derived from that effect algebra by the condition (GDE). Let $\mathbb{P} = P \cup P^*$ and $(\mathbb{P}; \ominus, 0, 1)$ be a D-algebra obtained by Proposition 2.6. Then P is an order ideal in the effect algebra $(\mathbb{P}; \oplus, 0, 1)$ derived from D-algebra $(\mathbb{P}; \ominus, 0, 1)$ by (GED) (under $\leq$ derived from $\oplus$), and for $a, b \in P$, $a \oplus_P b$ is defined in P iff $a \oplus b$ is defined in $\mathbb{P}$ and $a \oplus b = a \oplus_P b$.

3. SUBALGEBRAS OF *D*-ALGEBRAS AND EFFECT ALGEBRAS

In view of the Sections 1 and 2 we may consider both a D-poset $(P; \ominus, 0, 1)$ and the derived effect algebra $(P; \oplus, 0, 1)$ (and conversely) as a set P with 0, 1, $\leq$, $\ominus$, and $\oplus$ satisfying all properties (Di)–(Dii), (Ai)–(Aiv), and (Ei)–(Eiv), and also conditions (GEi)–(GEv), (GDi)–(GDvii), and (GED), (GDF) are satisfied. On the other hand, a D-algebra with a fundamental operation $\ominus$ and derived $\oplus$ and the (derived) effect algebra with the fundamental operation $\oplus$ and derived $\ominus$ are different from some algebraic points of view, e.g., they have different sets of subalgebras.

Definition 3.1. (1) For a generalized D-algebra $(P; \ominus, 0)$ a set $\emptyset \neq Q \subseteq P$ is called a *subalgebra* if $0 \in Q$, and for all $a, b \in Q$, if $b \ominus a$ is defined in P, then $b \ominus a \in Q$.

(2) For a generalized effect algebra $(P; \oplus, 0)$ a set $\emptyset \neq Q \subseteq P$ is called a *subalgebra* if $0 \in Q$, and for all $a, b \in Q$, if $a \oplus b$ is defined in P, then $a \oplus b \in Q$.

For subalgebras of D-algebras and effect algebras we assume in addition that $1 \in Q$.

Example 3.2. Suppose that $P = [0, 1]$ is the interval of real numbers with usual $\leq$, $+$, and $-$. Let us define partial binary operations $\ominus$ and $\oplus$ for all $a, b \in P$ as follows:

$b \ominus a$ is defined iff $a \leq b$, in which case $b \ominus a = b - a$.
$a \oplus b$ is defined iff $0 \leq a + b \leq 1$, in which case $a \oplus b = a + b$.

Then $(P; \leq, \ominus, 0, 1)$ is a D-poset and $(P; \oplus, 0, 1)$ is an effect algebra (derived from the D-poset). Let $Q = \{0\} \cup [\frac{1}{2}, 1]$. Then $(Q; \oplus, 0, 1)$ is a subalgebra of $(P; \oplus, 0, 1)$ but $(Q; \ominus, 0, 1)$ is not a subalgebra of $(P; \ominus, 0, 1)$ since $1 \ominus \frac{3}{4} = \frac{1}{4} \notin Q$.

Moreover, for $\frac{3}{4}, 1 \in Q$ we have $\frac{3}{4} \leq 1$ in P, but $\frac{3}{4} \not\leq 1$ in Q, since there does not exist $a \in Q$ with $\frac{3}{4} \oplus a = 1$. It also implies that $(Q; \ominus, 0, 1)$ is not an effect algebra in its own right.

We can easily show the following:

(i) If $(Q; \leq, \ominus, 0, 1)$ is a subalgebra of a D-poset $(P; \leq, \ominus, 0, 1)$, then it is a D-poset in its own right. Moreover, the partial order in Q is inherited from P (i.e., for all $a, b \in Q$, $a \leq b$ in Q iff $a \leq b$ in P).

(2) If $(A; \oplus, 0, 1)$ is a subalgebra of an effect algebra $(E; \oplus, \ominus, 1)$, then it need not be an effect algebra in its own right.

Example 3.3. The set $R^+ = [0, \infty)$ of nonnegative real numbers with usual $\leq$, $-$, and $+$ can be organized into a generalized D-algebra $(R^+; \ominus, 0)$ if we define for all $a, b \in R^+$:

$b \ominus a$ is defined iff $a \leq b$, in which case $b \ominus a = b - a$. The operation $\oplus$ derived from $\ominus$ on R^+ is total. The set $Q_1 = [0, 1]$ is a subalgebra of $(R^+; \ominus, 0)$, but it is not a subalgebra of $(R^+; \oplus, 0)$. since $1, \frac{1}{2} \in Q_1$, but $1 \oplus \frac{1}{2} \notin Q_1$. Moreover, e.g., $Q_2 = \{0\} \cup [\frac{1}{2}, \infty)$ is a subalgebra of $(R^+; \oplus, 0)$, but Q_2 is not a subalgebra of $(R^+; \ominus, 0)$. We see that $(Q_2; \oplus, 0)$ with the operation $\oplus$ inherited from $(R^+, \oplus, 0)$ is a generalized effect algebra in its own right, but the partial order in Q_2 is not inherited from P. For instance, $\frac{3}{4} \leq 1$ in R^+, while $\frac{3}{4} \not\leq 1$ in Q_2, since there is no $a \in Q_2$ with $\frac{3}{4} \oplus a = 1$.

The following proposition follows from Proposition 2.3.

Proposition 3.4. Let $(P; \oplus, 0)$ and $(P; \ominus, 0)$ be a generalized effect algebra and a generalized D-algebra derived one from the other. Let $\emptyset \neq Q \subseteq P$. Then $(Q; \oplus, 0)$ and $(Q; \ominus, 0)$ with inherited operations are a generalized effect algebra and a generalized D algebra in their own right derived one from the other and partial order in Q is inherited from P if and only if the following condition is satisfied

(S) If from elements $a, b, c \in P$ with $a \oplus b = c$ (or equivalently $c \ominus b = a$) at least two are elements of Q, then $a, b, c \in Q$.

For effect algebras and D-algebras we assume in addition that $1 \in Q$.

Greechie *et al.* (1995) define a *sub-effect algebra* of an effect algebra $(E; \oplus, 0, 1)$ as a subset F of E with properties (i) $0, 1 \in F$, (ii) $a \in F \Rightarrow a' = 1 \ominus a \in F$, and (iii) $a, b \in F$ with defined $a \oplus b$ in E implies $a \oplus b \in F$.

Proposition 3.5. (1) A set $F \subseteq E$ with $1 \in F$ is a sub-effect algebra of an effect algebra $(E; \oplus, 0, 1)$ iff F satisfies condition (S).

(2) A set $Q \subseteq P$ with $1 \in Q$ for a D-algebra $(P; \ominus, 0, 1)$ satisfies condition (S) iff Q is a subalgebra.

Assertion (2) of Proposition 3.5 fails to be true for effect algebras (Example 3.2).

Definition 3.6. A set $Q \subseteq P$ is a *sub-generalized effect algebra* of a generalized effect algebra $(P, \ominus, 0)$ if $0 \in Q$ and Q satisfies condition (S).

4. INTERVALS AND IDEAL SUBALGEBRAS OF GENERALIZED EFFECT ALGEBRAS

Definition 4.1. A subalgebra $Q \subseteq P$ of a generalized effect algebra $(P; \oplus, 0)$ is called an *ideal subalgebra* of P if it is an order ideal of P (i.e., $x \in Q$ and $y \leq x$ imply $y \in Q$).

Note that every ideal subalgebra Q of a generalized effect algebra $(P; \oplus, 0)$ satisfies condition (S); equally, for $a, b, c \in P$ if $a \oplus b = c$, then $c \in Q$ iff $a, b \in Q$.

Suppose that $(P; \oplus, 0)$ is a generalized effect algebra. For any $0 \neq w \in P$, the interval $[0, w] = \{x \in P | 0 \leq x \leq w\}$, under the partially defined operation obtained by restriction of $\oplus$ to $[0, w]$ (i.e., for $a, b \in [0, w]$, $a \oplus b$ in $[0, w]$ is defined iff $a \oplus b$ exists in P with $a \oplus b \leq w$), is an effect algebra with unit w. Evidently, for $a, b \in [0, w]$ the infimum $a \wedge_w b$ in $[0, w]$ exists iff $a \wedge b$ exists in P. On the other hand, the supremum $a \vee_w b$ in $[0, w]$ may exist when $a \vee b$ fails to exist in P. Nevertheless, if $x \vee y$ exists in P, then $a \vee_w b = a \vee b$.

Proposition 4.2. If for elements x, y of a generalized effect algebra $(P; \oplus, 0)$ there exist $x \oplus y$ and $x \vee y$, then $x \wedge y$ exists and

$$x \oplus y = (x \vee y) \oplus (x \wedge y)$$

Proof. Let us put $w = x \oplus y$ and consider the effect algebra $[0, w]$. Then $x \vee_w y = x \vee y \leq w$ and $w = x \oplus y \in [0, w]$. Thus, by Theorem 3.5 of Greechie *et al.* (1995), $(x \vee_w y) \oplus (x \wedge_w y) = x \oplus w$ in $[0, w]$. Hence we have $(x \vee y) \oplus (x \wedge y) = x \oplus w$ in P.

Corollary 4.3. In every lattice ordered generalized effect algebra $(P; \oplus, 0)$

$$a \oplus b = (a \vee b) \oplus (a \wedge b)$$

for all $a, b \in P$ with defined $a \oplus b$.

The following example shows that an interval in a generalized effect algebra $(P; \ominus, 0)$ need not be a sub-generalized effect algebra of P and a sub-generalized effect algebra of P need not be an ideal subalgebra of P.

Example 4.4. Let on the set $P = \{0, a, b, c, d\}$ the partial binary operation $\oplus$ be defined as follows: $a \oplus b = c$, $a \oplus a = b \oplus b = d$, and $x \ominus 0 = x$ for all $x \in P$. Then $(P; \oplus, 0)$ is a generalized effect algebra. Evidently, $a, b \leq c$ and $a, b \leq d$. Moreover, $0 \leq x$ for all $x \in P$. The interval $[0, c]$ is an effect algebra with unit c, but it is not a sub-generalized effect algebra of $(P; \oplus, 0)$ (even, $[0, c]$ is not a subalgebra of P) since $a \oplus a \notin [0, c]$. On the other hand, if $S = \{0, a, d\}$, then $(S; \oplus, 0)$ is a sub-generalized effect algebra of P which is not an ideal subalgebra of P.

Proposition 4.5. For a generalized effect algebra $(P; \oplus, 0)$ and an interval $[0, w]$ $(0 \neq w \in P)$ the following conditions are equivalent:

(i) $[0, w]$ is a subalgebra of $(P; \oplus, 0)$.
(ii) $[0, w]$ is a sub-generalized effect algebra of $(P; \oplus, 0)$.
(iii) $[0, w]$ is an ideal subalgebra of $(P; \oplus, 0)$.

Proof. The statement follows from the fact that for $x, y \in P$ with $x \oplus y$ defined we have $x, y \leq x \oplus y$.

Definition 4.6. We say that an element $w \neq 0$ of a generalized effect algebra $(P; \oplus, 0)$ has the property (IS) if for all $x \in P$:

$$x \wedge w \text{ exists and } x = (x \wedge w) \vee (x' \oplus (x \wedge w))$$

Theorem 4.7. Let the element $w \neq 0$ of a generalized effect algebra $(P; \oplus, 0)$ has the property (IS). Then for every $x \in P$:

(i) $(x \ominus (x \wedge w)) \wedge w = 0$.
(ii) If $x \oplus w$ is defined, then $x \vee w$ and $x \wedge w$ exist and $x \vee w = x \oplus w$ and $x \wedge w = 0$.
(iii) $[0, w]$ is a sub-generalized effect algebra of $(P; \oplus, 0)$; hence $[0, w]$ is an ideal subalgebra of $(P; \oplus, 0)$.

Proof. (i) By (IS) $x = (x \wedge w) \vee (x \ominus (x \wedge w)) = (x \wedge w) \oplus (x \ominus (x \wedge w))$. It follows that $(x \wedge w) \wedge (x \ominus (x \wedge w)) = 0$, using Proposition 4.2 and the cancellation law. Thus $w \wedge (x \ominus (x \wedge w)) = 0$.

(ii) If $x \oplus w$ exists, then $(x \oplus w) \wedge w = w$ and $(x \oplus w) \ominus w = x$ and hence $x \oplus w = w \vee x$ by (IS). In view of Proposition 4.2 and the cancellation property we obtain $x \wedge w = 0$.

(iii) Suppose that $x, y \in [0, w]$ with $x \oplus y$ defined in P. Then by (IS) $x \oplus y = ((x \oplus y) \wedge w) \vee ((x \oplus y) \ominus (x \oplus y) \wedge w) \leq ((x \oplus y) \wedge w) \vee ((x \oplus y) \ominus y) = ((x \oplus y) \wedge w) \vee x \leq w$ since $y \leq (x \oplus y) \wedge w$. Thus $[0, w]$ is a subalgebra of P. In view of Proposition 4.6 we obtain the statement (iii).

5. CENTRAL ELEMENTS OF GENERALIZED EFFECT ALGEBRAS

In Greechie *et al.* (1995) the notion of a central element of an effect algebra was introduced and the decompositions of effect algebras into direct products of ideal subalgebras was studied.

A direct product of two generalized effect algebras $(P_1; \oplus_1, 0_1)$ and $(P_2; \oplus_2, 0_2)$ is the generalized effect algebra $(P; \oplus, 0)$, where $P = P_1 \times P_2$, the partial binary operation $\oplus$ is defined coordinatewise, and 0 is the couple $(0_1, 0_2)$. Obviously, the partial order in P is also defined coordinatewise. We can easily see that P is an effect algebra iff P_1 and P_2 are effect algebras under which $1 \in P$ is the couple $(1_1, 1_2)$ (see Proposition 2.5).

Definition 5.1. For a generalized effect algebra $(P; \oplus, 0)$ an element $z \in P$ is called a *central element* iff for all $x, y \in P$ the following conditions are satisfied:

(Ci) $x \wedge z$ exists and $x = (x \wedge z) \vee (x \ominus (x \wedge z))$ [i.e., z satisfies (IS)].
(Cii) If $x \wedge z = 0$, then $x \oplus z$ is defined.
(Ciii) If $x \oplus y$ is defined and $x \wedge z = y \wedge z = 0$, then $(x \oplus y) \wedge z = 0$.

Theorem 5.2. Let z be a central element of a generalized effect algebra $(P; \oplus, 0)$ and $Q_z = \{x \in P | x \wedge z = 0\}$. Then $[0, z]$ and Q_z are ideal subalgebras of $(P; \oplus, 0)$. Moreover, $[0, z]$ and $[0, z^*]$ are ideal subalgebras of the effect algebra $(\mathbb{P}; \oplus, 0, 1)$, where $\mathbb{P} = P \dot{\cup} P^*$ is obtained by Proposition 2.6 and $Q_z = [0, z^*] \cap P$.

Proof. $[0, z]$ is an ideal subalgebra of $(P; \oplus, 0)$ in view of Theorem 4.7. Let $x, y \in Q_z$ with defined $x \oplus y$. Then $x \wedge z = y \wedge z = 0$ and by (Ciii) also $(x \oplus y) \wedge z = 0$. Hence $x \oplus y \in Q_z$. Evidently Q_z is an order ideal of P. Thus Q_z is an ideal subalgebra of $(P; \oplus, 0)$. By condition (Cii) and by Theorem 4.7 we obtain $Q_z = P \cap [0, z^*]$. Since $[0, z]$ is an ideal subalgebra of P and P is an ideal subalgebra of $\mathbb{P}$, we conclude that $[0, z]$ is an ideal subalgebra of $\mathbb{P}$.

Let us show now that $[0, z^*]$ is an ideal subalgebra of $\mathbb{P}$. If $x, y \in [0, z^*] \cap P$ with defined $x \oplus y$, then $x \oplus y \in Q_z \subseteq [0, z^*]$. If $x, y \in [0, z^*] \cap P^*$, then $x \oplus y$ is not defined, since $x, y \leq x \oplus y$ implies $x \oplus y \in P^*$ and hence $x = (x \oplus y) \ominus y \in P$ (see Proposition 2.6), a contradiction. If $x \in [0, z^*] \cap P$, $y \in [0, z^*] \cap P^*$, and $x \oplus y$ is defined, then $x \leq 1 \ominus z$, $x \leq$

$1 \ominus y$, and $z \leq 1 \ominus y$. By Theorem 4.7, (ii) $x \oplus z = x \vee z$ and hence $x \oplus z \leq 1 \ominus y$. It follows that $(x \oplus y) \oplus z$ is defined and hence $x \oplus y \in [0, z^*]$.

In Greechie *et al.* (1995) an element z of an effect algebra $(E; \oplus, 0, 1)$ is called *central* iff (1) for every $x \in E$ there exist $u \leq z$, $v \leq 1 \ominus z$ such that $x = u \oplus v$ and (2) $[0, z]$ and $[0, 1 \ominus z]$ are ideal subalgebras of E.

Theorem 5.3. An element $z \in P$ is a central element of a generalized effect algebra $(P; \oplus, 0)$ iff z is a central element of the effect algebra $(\mathbb{P}; \oplus, 0, 1)$, where $\mathbb{P} = P \cup P^*$ is obtained by Proposition 2.6.

Proof. (1) Suppose that $z \in P$ is a central element of $(P; \oplus, 0)$. If $x \in P$, then $x = (x \wedge z) \oplus (x \ominus (x \wedge z))$. By Theorem 4.7(i), $(x \ominus (x \wedge z)) \wedge z = 0$, and by the property (Cii) of central elements, $(x \ominus (x \wedge z)) \oplus z$ is defined. Thus $x \wedge z \leq z$ and $x \ominus (x \wedge z) \leq 1 \ominus z$. If $x^* \in P^*$, then $1 \ominus x^* = x \in P$ and hence $x = (x \wedge z) \vee (x \ominus (x \wedge z))$, where $x \ominus (x \wedge z) \leq 1 \ominus z$. It follows that $x^* = 1 \ominus x = (1 \ominus (x \ominus (x \wedge z))) \ominus (x \wedge z) \geq z \ominus (x \wedge z)$. By Proposition 2.6, $x^* \ominus (z \ominus (x \wedge z)) = (x \oplus (z \ominus (x \wedge z)))^* = ((x \ominus (x \wedge z)) \oplus (x \wedge z) \oplus (z \ominus (x \wedge z))^* = ((x \ominus (x \wedge z)) \oplus z)^* \leq z^* = 1 \ominus z$. Thus $x^* = (z \ominus (x \wedge z)) \oplus (x^* \ominus (z \ominus (x \wedge z))$, where $z \ominus (x \wedge z) \leq z$ and $x^* \ominus (z \ominus (x \wedge z)) \leq 1 \ominus z$. By Theorem 5.2, $[0, z]$ and $[0, z^*]$ are ideal subalgebras of the effect algebra $(P; \oplus, 0, 1)$. We obtain that z is a central element of $(\mathbb{P}; \oplus, 0, 1)$.

(2) Suppose that $z \in P$ and z is a central element of $(\mathbb{P}; \oplus, 0, 1)$. Then $(\mathbb{P}; \oplus, 0, 1)$ is isomorphic to the direct product of effect algebras $[0, z]$ and $[0, z^*]$ (meaning with inherited operations $\oplus$ from $\mathbb{P}$) and $x = (x \wedge z) \oplus (x \wedge z^*)$ for every $x \in \mathbb{P}$ (Greechie *et al.* 1995). It follows that $x \wedge z^* = x \ominus (x \wedge z)$. By the definition of the Cartesian product we have also $x = (x \wedge z) \vee (x \wedge z^*)$, for every $x \in \mathbb{P}$. We obtain that $x \wedge z = 0$ iff $x \in [0, z^*]$ is an ideal subalgebra of $\mathbb{P}$, z satisfies also condition (Ciii) of Definition 5.1.

Theorem 5.4. An element $z \in P$ of a generalized effect algebra $(P; \oplus, 0)$ is central if and only if P is isomorphic to a direct product of $[0, z]$ and $Q_z = \{x \in P | x \wedge z = 0\}$ meaning with operations $\oplus$ restricted from P.

Proof. This follows by Theorem 5.3 and Greechie *et al.* (1995), using the fact that if $u \in [0, z]$, $v \in [0, z^*]$ are such that $u \oplus v \in P$, then $v \in P$ by Proposition 2.6.

Theorem 5.5. An element $z \in E$ is a central element of an effect algebra $(E; \oplus, 0, 1)$ iff for every $x \in E$, $x \wedge z$ and $x \wedge z'$ exist and $x = (x \wedge z) \vee (x \wedge z')$.

Proof. (1) If z is a central element of E (in the sense of Greechie *et al.*, 1995) then for every $x \in E$ we have $x = (x \wedge z) \oplus (x \wedge z') = (x \wedge z) \vee$

$(x \wedge z')$, in view of Greechie *et al.* (1995) and the definition of the Cartesian product $[0, z] \times [0, z']$.

(2) Suppose that for every $x \in E$ there exist $x \wedge z$, $x \wedge z'$, and $x = (x \wedge z) \vee (x \wedge z')$. Then $1 = z \vee z'$. Since $1 = z \oplus z'$, we obtain $z \wedge z' = 0$. It follows that for every $x \in E$ there exists $(x \wedge z) \oplus (x \wedge z')$ and $(x \wedge z) \wedge (x \wedge z') = 0$. Hence $x = (x \wedge z) \oplus (x \wedge z')$, which implies that $x \wedge z = x \ominus (x \wedge z')$ and $(x \wedge z) \vee (x \ominus (x \wedge z)) = (x \wedge z') \vee (x \ominus (x \wedge z'))$.

By Theorem 4.7, $[0, z]$ and $[0, z']$ are ideal subalgebras of E. We conclude that z and z' are central elements of E in the sense of Greechie *et al.* (1995).

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